Math 2002 Final

No calculators or other aids are allowed. Be sure to write your name and student ID# on the booklet provided. Questions 2 and 3 are worth 4 points, and all other questions are worth 6. You have three hours. Good luck!

1. (a) State Fubini's theorem for double integrals.
   (b) State the fundamental theorem of line integrals.
   (c) State Stokes' theorem.

2. Evaluate the following integrals:
   (a) The triple integral
   $$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx$$
   (b) The line integral
   $$\int_C y^2 + e^x \, dx$$
   where C is the line from $(-1, 2)$ to $(-3, 5)$.

3. Solve the initial value problem
   $$y'' + 6y' + 9y = 3x + 1$$
   assuming that $y(0) = 0, \; y'(0) = 1$.

4. For the vector field $\mathbf{F} = (\cos y)\mathbf{i} + (z^2 \cos y - x \sin y)\mathbf{j} + (2z \sin y + 3)\mathbf{k}$,
   (a) Show that $\mathbf{F}$ is conservative.
   (b) Find a function f so that $\triangledown f = \mathbf{F}$.
   (c) Calculate how much work $\mathbf{F}$ does as a particle moves along the curve C, where C has parametrization $r(t) = (t^2, \pi t, t^2 + 1)$ and $0 \le t \le 1$.

5. Evaluate the line integral $\int_C \mathbf{F} \cdot dr$, where $\mathbf{F} = (x^2 + 2y)\mathbf{i} + (6x + y^3)\mathbf{j}$, and C travels along the line from $(0, 0)$ to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then from $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ to $(1, 0)$ along the curve $x^2 + y^2 = 1$, and finally along the line from $(1, 0)$ to $(0, 0)$.

6. Use the change of variables $u = \frac{y}{2}$, $v = x - 2y$ to change the double integral

$$\int_D \sqrt{x-2y} + \frac{y^2}{4}\ dA$$

where D is the triangle with vertices $(0,0), (4,0), (4,2)$. You do not need to evaluate the integral, only set it up (including the bounds of integration).

7. Suppose that S is a surface consisting of two parts: the first part is $z = x^2 + y^2$ where $z \leq 4$, and the second part is the intersection of the curves $z = 4$ and $x^2 + y^2 = 4$. If we also have

$$\mathbf{F} = (2x + y^2)\mathbf{i} + (z^4)\mathbf{j} + (3z + y^2x^2)\mathbf{k},$$

then calculate the flow of $\mathbf{F}$ past the surface S.

8. Find the general solution of the differential equation

$$y'' - y = e^x \cos x$$

using either the method of undetermined coefficients or the method of variation of parameters.

9. Find the value of

$$\int_S \mathrm{curl}(\mathbf{F}) \cdot\ d\mathbf{S}$$

where

$$\mathbf{F} = x^4\mathbf{i} + \left(\frac{x^2}{2} + \cos y + e^z\right)\mathbf{j} + (3y + z^3)\mathbf{k}$$

and S is the part of the surface $y = \sqrt{x^2 + z^2}$ that intersects the cylinder $x^2 + z^2 = 4$, oriented towards the origin.

10. **(Bonus question, +4 marks)** Prove that for every continuous function $f : R^3 \to R$ there exists some vector field $\mathbf{F}$ such that $\mathrm{div}(\mathbf{F}) = f$.

If you need it: **Spherical co-ordinates:**

$$x = r \sin\phi \cos\theta,\ \ y = r \sin\phi \sin\theta,\ \ z = r\cos\phi$$

the Jacobian for the spherical co-ordinates transformation is $r^2 \sin\phi$.